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GENERALIZED FRACTIONAL PROGRAMMING

J. C. LIU
Section of Mathematics, National Overseas Chinese Student University,
PO Box 1-1337 Linkou, 24499, Taiwan.

Y. KIMURA
Department of Mathematics and Information Science,
Graduate School of Science and Technology,
Niigata University 950-21, Niigata, Japan.

and

K. TANAKA
Department of Mathematics, Niigata University,
950-21, Niigata, Japan.

Optimality conditions in generalized fractional programming involving nonsmooth Lipschitz functions are established. Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models, and several duality theorems are derived.

KEY WORDS: Generalized fractional programming, invex, quasiinvex, pesudoinvex, duality.

1. INTRODUCTION

In this paper, we consider the following minimax fractional programming problem:

$$(P) \qquad v^* = \min_{x\in S}\max_{1\leq i\leq p}[f_i(x)/g_i(x)],$$

where

(A1) $S=\{x\in\mathbb{R}^n; h_k(x)\leq 0, k=1,2,\cdots,m\}$ is nonempty and compact;
(A2) $f_i : X_0\mapsto\mathbb{R}, g_i : X_0\mapsto\mathbb{R}, i=1,2,\cdots,p$, and $h_k : X_0\mapsto\mathbb{R}, k=1,2,\cdots,m$ are locally Lipschitz continuous and X_0 is the open subset of $\mathbb{R}^n$;
(A3) $g_i(x)>0, i=1,2,\cdots,p,\ x\in S$;
(A4) if g_i is not affine, then $f_i(x)\geq 0$ for all i and all $x\in S$.

Generalized fractional programming has been of much interest in the last decades; see for example [1-4, 6, 7, 10-19]. In [7], Crouzeix *et al.* have shown that the minimax fractional program can be derived by solving the following minimax nonlinear (nondifferentiable) parametric program:

$$(P_v) \qquad \min_{x\in S}\max_{1\leq i\leq p}\big(f_i(x)-vg_i(x)\big)$$

where $v\in\mathbb{R}_+\equiv[0,\infty)$ is a parameter.

It is clear that (P_v) is equivalent to the following problem (EP_v) for a given v:

$$\begin{array}{lll}(EP_v) & \min q, & \\ & \text{subject to} & f_i(x) - vg_i(x) \leq q, \quad i = 1, 2, \cdots, p, \\ & & h_k(x) \leq 0, \quad k = 1, 2, \cdots, m.\end{array}$$

In [2], Bector *et al.* employed the problem (EP_v) to prove necessary and sufficient optimality conditions for problem (P) and establish various duality results for problem (EP_v) involving differentiable generalized convex functions (or generalized invex functions). Liu [10-12] also adapted the same approach to obtain necessary and sufficient optimality conditions; and he derived duality theorems for generalized fractional programming problems involving either nonsmooth pseudoinvex functions [11] or nonsmooth (F, ρ)-convex functions [10], and duality theorems for generalized fractional variational problems involving generalized (F, ρ)-convex functions [12].

But, all of the above necessary optimality conditions and strong duality theorems need that the constraint of (EP_v) satisfy a constraint qualification.

In order to improve this defect, we want to use problem (P_v) to establish both parametric and nonparameter necessary and sufficient optimality conditions, since a constraint qualification that is imposed on the constrains of (P) may not hold for (EP_v) but hold for (P_v). Subsequently, these optimality criteria are utilized as a basis for constructing one parametric and two other parametric-free dual models (see [13] and [16]), and some duality results for (P) are established.

2. NOTATIONS AND PRELIMINARY RESULTS

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be its non-negative orthant. Let X_0 be an open subset of $\mathbb{R}^n$.

Definition 2.1. The function $\theta : X_0 \mapsto \mathbb{R}$ is said to be **Lipschitz** on X_0 if there exists $c > 0$ such that for all $y, x \in X_0$,

$$|\theta(y) - \theta(x)| \leq c\|y - x\|,$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

For each d in $\mathbb{R}^n$, $\theta^\circ(x; d)$ is the **generalized directional derivative of Clarke** [5] defined by

$$\theta^\circ(x; d) = \limsup_{\substack{y \to x \\ t \downarrow 0}} [\theta(y + td) - \theta(y)]/t.$$

It then follows that

$$\theta^\circ(x; d) = \max\{\xi^T d \mid \xi \in \partial\theta(x)\} \qquad \text{for any } x \text{ and } d,$$

where $\partial\theta(\cdot)$ denotes the **Clarke's generalized gradient** [5]. The following definitions can be found in [11]:

Definition 2.2. The function $\theta : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be **invex** at x^* with respect to η if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta(x) - \theta(x^*) \geq \theta^\circ(x^*; \eta(x, x^*)). \tag{2.1}$$

θ is said to be invex on $\mathbb{R}^n$ with respect to η if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta(x) - \theta(u) \geq \theta^\circ(u; \eta(x, u)). \tag{2.2}$$

If we have strict inequality in (2.1) and (2.2), respectively, then θ is said to be **strictly invex** at x^* with respect to η and strictly invex on $\mathbb{R}^n$ with respect to η, respectively.

Definition 2.3. The function $\theta : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be **quasiinvex** at x^* with respect to η if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta(x) \leq \theta(x^*) \Rightarrow \theta^\circ(x^*; \eta(x, x^*)) \leq 0. \tag{2.3}$$

θ is said to be quasiinvex on $\mathbb{R}^n$ with respect to η if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta(x) \leq \theta(u) \Rightarrow \theta^\circ(u; \eta(x, u)) \leq 0. \tag{2.4}$$

If we have strict inequality in (2.3) and (2.4), respectively, then θ is said to be **strictly quasiinvex** at x^* with respect to η and strictly quasiinvex on $\mathbb{R}^n$ with respect to η, respectively.

Definition 2.4. The function $\theta : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be **pseudoinvex** at x^* with respect to η if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that, for each $x \in \mathbb{R}^n$,

$$\theta^\circ(x^*; \eta(x, x^*)) \geq 0 \Rightarrow \theta(x) \geq \theta(x^*). \tag{2.5}$$

θ is said to be pseudoinvex on $\mathbb{R}^n$ with respect to η if there exists a mapping $\eta : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ such that, for each $x, u \in \mathbb{R}^n$,

$$\theta^\circ(u; \eta(x, u)) \geq 0 \Rightarrow \theta(x) \geq \theta(u). \tag{2.6}$$

If we have strict inequality in (2.5) and (2.6), respectively, then θ is said to be **strictly pseudoinvex** at x^* with respect to η and strictly pseudoinvex on $\mathbb{R}^n$ with respect to η, respectively.

We need the following lemmas.

Lemma 2.1. [16, Lemma 3.1.] Let v^* be the optimal value of (P), and let $V(v)$ be the optimal value of (P_v) for any fixed $v \in \mathbb{R}_+$ such that (P_v) has an optimal solution. Then x^* is an optimal solution of (P) if and only if x^* is an optimal solution of (P_{v^*}) with optimal value $V(v^*) = 0$.

Lemma 2.2. [5, Proposition 2.3.12.] Let $f_1, \cdots, f_p$ be Lipschitz functions at x^* and $\alpha_i \in \mathbb{R}$ for all $i = 1, \cdots, p$. Then

(1) $\partial(\sum_{i=1}^{p} \alpha_i f_i)(x^*) \subset \sum_{i=1}^{p} \alpha_i \partial f_i(x^*)$,

(2) $\partial\big[\max_{1\le i\le p} f_i\big](x^*) \subset \bigcup \big\{ \sum_{l\in L} \alpha_l \partial f_l(x^*);\ \alpha_l \ge 0, \sum_{l\in L} \alpha_l = 1 \big\}$ where L is the set of indices l for which

$$f_l(x^*) = \max_{1\le i\le p} f_i(x^*).$$

Lemma 2.3. [16, Lemma 3.2.] For each $x \in S$, one has

$$\phi(x) \equiv \max_{1\le i\le p} \big(f_i(x)/g_i(x)\big) = \max_{\beta\in U}\Big(\sum_{i=1}^{p} \beta_i f_i(x) / \sum_{i=1}^{p} \beta_i g_i(x)\Big)$$

where $U = \{\beta \in \mathbb{R}_+^p \,|\, \sum_{i=1}^{p} \beta_i = 1\}$.

For convenience, we give the scalar minimization problem as follows:

$$\begin{aligned} (SP) \quad & \text{Minimize} \quad N(x), \\ & \text{subject to} \quad h_k(x) \le 0, \quad k = 1, 2, \cdots, m \end{aligned}$$

where $N, h_k : X_0 \mapsto \mathbb{R}, k = 1, 2, \cdots, m$, are Lipschitz on X_0. We need the following lemma.

Lemma 2.4. [8, Theorem 6.] If $x^* \in X_0$ is a local minimum for (SP) and a constraint qualification is satisfied, then there exist $z^* = (z_1^*, \cdots, z_m^*) \in \mathbb{R}_+^m$ such that

$$\begin{aligned} & 0 \in \partial N(x^*) + \sum_{k=1}^{m} z_k^* \partial h_k(x^*), \\ & z_k^* h_k(x^*) = 0, \qquad \text{for all} \quad k = 1, 2, \cdots, m. \end{aligned}$$

For simplicity, throughout the paper we denote

$$\begin{aligned} & U = \{\alpha \in \mathbb{R}_+^p \mid \sum_{i=1}^{p} \alpha_i = 1\}, \\ & F(x) = (f_1(x), \cdots, f_p(x)), \\ & G(x) = (g_1(x), \cdots, g_p(x)), \quad \text{and} \\ & H(x) = (h_1(x), \cdots, h_m(x)). \end{aligned}$$

For $z \in \mathbb{R}^m$, $z^\top H(x^*) = \sum_{k=1}^{m} z_k h_k(x^*)$, and $\partial(z^\top H)(x^*) = \sum_{k=1}^{m} z_k \partial h_k(x^*)$.

3. NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS

In this section, we shall use Lemmas 2.1 $\sim$ 2.4 to establish some necessary and sufficient optimality conditions for the minimax fractional programming problem (P).

Theorem 3.1 (Necessary optimality conditions). Let $x^* \in S$. If x^* is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $v^* = \phi(x^*) \in \mathbb{R}_+$, $y^* \in U$, $z^* \in \mathbb{R}_+^m$ such that

$$0 \in \partial({y^*}^\top F)(x^*) - v^* \partial({y^*}^\top G)(x^*) + \partial({z^*}^\top H)(x^*), \tag{3.1}$$

$${y^*}^\top F(x^*) - v^* {y^*}^\top G(x^*) = 0, \tag{3.2}$$

$${z^*}^\top H(x^*) = 0. \tag{3.3}$$

Proof. If x^* is an optimal solution of (P), by Lemma 2.1, it is an optimal solution of (P_{v^*}) with $v^* = \max_{1 \le i \le p}[f_i(x^*)/g_i(x^*)]$. Thus, by Lemma 2.4, there exist $z^* \in \mathbb{R}_+^m$, such that

$$0 \in \partial\Big(\max_{1 \le i \le p}(f_i - v^* g_i)\Big)(x^*) + \partial({z^*}^\top H)(x^*)$$

and

$${z^*}^\top H(x^*) = 0.$$

Therefore, by Lemma 2.2, there exist $\alpha_l \ge 0$, $l \in L$, $\sum_{l \in L} \alpha_l = 1$, such that

$$0 \in \sum_{l \in L} \alpha_l \big(\partial f_l(x^*) + v^* \partial(-g_l(x^*))\big) + \partial({z^*}^\top H)(x^*). \tag{3.4}$$

It is obvious that $v^* = \max_{1 \le i \le p}[f_i(x^*)/g_i(x^*)]$ if and only if $\max_{1 \le i \le p}[f_i(x^*) - v^* g_i(x^*)] = 0$. From (3.4), if we set $y_i^* = \alpha_i$ for $i \in L$ as well as $y_i^* = 0$ for $i \in \{1, 2, \cdots, p\} \setminus L$, the expressions (3.1), (3.2) and (3.3) hold.

□

In order to construct parameter-free duality models for problem (P), we shall formulate parameter-free versions of Theorem 3.1 as follows:

Theorem 3.2. Let $x^* \in S$. If x^* is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$ and $z^* \in \mathbb{R}_+^m$ such that

$$0 \in {y^*}^\top G(x^*)\Big(\partial({y^*}^\top F)(x^*) + \partial({z^*}^\top H)(x^*)\Big) - {y^*}^\top F(x^*) \partial({y^*}^\top G)(x^*), \tag{3.5}$$

$${z^*}^\top H(x^*) = 0, \tag{3.6}$$

and obtain the optimal value by

$$\phi(x^*) = {y^*}^\top F(x^*)/{y^*}^\top G(x^*) = \max_{1 \le i \le p}\big(f_i(x^*)/g_i(x^*)\big). \tag{3.7}$$

Proof. From (3.2) and (3.1), substituting ${y^*}^\top F(x^*)/{y^*}^\top G(x^*)$ for v^*, we can derive the results.

□

The conditions (3.5) $\sim$ (3.7) will be the sufficient optimality condition which we state as the following theorem.

Theorem 3.3 (Sufficient optimality conditions). Let $x^* \in S$, and assume that there exist $y^* \in U$ and $z^* \in \mathbb{R}_+^m$, such that the conditions (3.5) $\sim$ (3.7) hold. Let

$$A(x) = {y^*}^\top G(x^*){y^*}^\top F(x) - {y^*}^\top F(x^*){y^*}^\top G(x),$$

$$B(x) = {z^*}^\top H(x), \qquad \text{and} \qquad C(x) = A(x) + {y^*}^\top G(x^*)B(x).$$

If any one of the following conditions holds

(a) A is pseudoinvex at x^* with respect to η and B is quasiinvex at x^* with respect to same function η,
(b) A is quasiinvex at x^* with respect to η and B is strictly pseudoinvex at x^* with respect to same function η,
(c) C is pseudoinvex at x^* with respect to η.

Then x^* is an optimal solution of (P).

Proof. Suppose contrary that x^* were not an optimal solution of (P). Then there exists a feasible solution $x_1 \in S$ such that

$$\phi(x^*) > \phi(x_1).$$

From (3.7) and Lemma 2.3, we have

$${y^*}^\top F(x^*)/{y^*}^\top G(x^*) > \max_{\beta \in U}(\beta^\top F(x_1)/\beta^\top G(x_1)) \geq {y^*}^\top F(x_1)/{y^*}^\top G(x_1).$$

It follows that

$$A(x_1) = {y^*}^\top G(x^*){y^*}^\top F(x_1) - {y^*}^\top F(x^*){y^*}^\top G(x_1) < 0 = A(x^*). \tag{3.8}$$

Using both the feasibility x_1 for (P) and the equality (3.6), we have

$$B(x_1) \leq 0 = B(x^*). \tag{3.9}$$

Consequently, expressions (3.8) and (3.9) yield

$$C(x_1) < C(x^*). \tag{3.10}$$

By (3.5), there exist $\xi \in \partial({y^*}^\top F)(x^*)$, $\zeta \in \partial({z^*}^\top H)(x^*)$, and $\rho \in \partial(-{y^*}^\top G)(x^*)$, such that

$${y^*}^\top G(x^*)(\xi + \zeta) + {y^*}^\top F(x^*)\rho = 0.$$

From here it results

$${y^*}^\top G(x^*)(\xi^\top \eta(x, x^*) + \zeta^\top \eta(x, x^*)) + {y^*}^\top F(x^*)\rho^\top \eta(x, x^*) = 0. \tag{3.11}$$

Using the characterization of the generalized gradient of Clarke, we obtain

$$({y^*}^\top F)^\circ(x^*; \eta(x, x^*)) \geq \xi^\top \eta(x, x^*), \quad \text{for all} \quad x \in S, \tag{3.12}$$

$$({z^*}^\top H)^\circ(x^*; \eta(x, x^*)) \geq \zeta^\top \eta(x, x^*), \quad \text{for all} \quad x \in S, \tag{3.13}$$

$$(-y^{*\top}G)^\circ(x^*;\eta(x,x^*)) \geq \rho^\top\eta(x,x^*), \quad \text{for all} \quad x \in S. \tag{3.14}$$

Now, multiplying (3.12) by $y^{*\top}G(x^*)$, (3.13) by $y^{*\top}G(x^*)$, and (3.14) by $y^{*\top}F(x^*)$, and adding the resulting inequalities and with (3.11), we obtain

$$\begin{aligned} y^{*\top}G(x^*)[(y^{*\top}F)^\circ(x^*;\eta(x,x^*)) + (z^{*\top}H)^\circ(x^*;\eta(x,x^*))] \\ - y^{*\top}F(x^*)(y^{*\top}G)^\circ(x^*;\eta(x,x^*)) \geq 0, \quad \text{for all} \quad x \in S. \end{aligned} \tag{3.15}$$

If hypothesis (a) holds, using the pseudoinvexity of A at x^* and the inequality (3.8), we have

$$y^{*\top}G(x^*)(y^{*\top}F)^\circ(x^*;\eta(x_1,x^*)) - y^{*\top}F(x^*)(y^{*\top}G)^\circ(x^*;\eta(x_1,x^*)) < 0. \tag{3.16}$$

Consequently, the inequalities (3.15) and (3.16) yield

$$y^{*\top}G(x^*)(z^{*\top}H)^\circ(x^*;\eta(x_1,x^*)) > 0.$$

Thus, we have

$$(z^{*\top}H)^\circ(x^*;\eta(x_1,x^*)) > 0. \tag{3.17}$$

Using the quasiinvexity of B at x^*, we get from (3.17)

$$B(x_1) = z^{*\top}H(x_1) > z^{*\top}H(x^*) = B(x^*)$$

which contradicts the inequality (3.9).

Hypothesis (b) follows along with the same lines as (a).

If hypothesis (c) holds, using the pseudoinvexity of C at x^* and the inequality (3.10), we have

$$\begin{aligned} y^{*\top}G(x^*)[(y^{*\top}F)^\circ(x^*;\eta(x_1,x^*)) + (z^{*\top}H)^\circ(x^*;\eta(x_1,x^*))] \\ - y^{*\top}F(x^*)(y^{*\top}G)^\circ(x^*;\eta(x_1,x^*)) < 0 \end{aligned}$$

which contradicts the inequality (3.15). Hence, the proof is complete. □

4. THE FIRST DUAL MODEL

Utilize Theorem 3.2, in Sections 4 and 5 we shall introduce two parametric-free dual models and prove appropriate duality theorems. Indeed, we shall demonstrate that the following is dual problem for (P):

$$\begin{aligned} (DI) \quad \text{Maximize} \quad & (y^\top F(u) + z^\top H(u))/y^\top G(u) \\ \text{subject to} \quad & 0 \in y^\top G(u)\big(\partial(y^\top F)(u) + \partial(z^\top H)(u)\big) \\ & \quad - (y^\top F(u) + z^\top H(u))\partial(y^\top G)(u), \qquad (4.1) \\ & y \in U, \; z \in \mathbb{R}^m_+. \qquad (4.2) \end{aligned}$$

We denote by K_1 the set of all feasible solutions $(u,y,z) \in X_0 \times U \times \mathbb{R}^m_+$ of problem (DI). We assume throughout this section that $y^\top F(u) + z^\top H(u) \geq 0$ and $y^\top G(u) > 0$.

Theorem 4.1 (Weak Duality). Let $x \in S$ and $(u, y, z) \in K_1$ and assume that

$$D(\cdot) = y^\top G(u)[y^\top F(\cdot) + z^\top H(\cdot)] - y^\top G(\cdot)[y^\top F(u) + z^\top H(u)]$$

is a pseudoinvex function with respect to η at u. Then

$$\phi(x) \geq \big(y^\top F(u) + z^\top H(u)\big)/y^\top G(u).$$

Proof. By (4.1), there exist $\xi \in \partial(y^\top F)(u)$, $\zeta \in \partial(z^\top H)(u)$, and $\rho \in \partial(-y^\top G)(u)$, such that

$$y^\top G(u)(\xi + \zeta) + [y^\top F(u) + z^\top H(u)]\rho = 0.$$

From here it results

$$y^\top G(u)(\xi^\top \eta(x,u) + \zeta^\top \eta(x,u)) + [y^\top F(u) + z^\top H(u)]\rho^\top \eta(x,u) = 0. \tag{4.3}$$

Using the characterization of the generalized gradient of Clarke, we obtain

$$(y^\top F)^\circ(u; \eta(x,u)) \geq \xi^\top \eta(x,u), \quad \text{for all} \quad x \in S, \tag{4.4}$$

$$(z^\top H)^\circ(u; \eta(x,u)) \geq \zeta^\top \eta(x,u), \quad \text{for all} \quad x \in S, \tag{4.5}$$

$$(-y^\top G)^\circ(u; \eta(x,u)) \geq \rho^\top \eta(x,u), \quad \text{for all} \quad x \in S. \tag{4.6}$$

Now, multiplying (4.4) by $y^\top G(u)$, (4.5) by $y^\top G(u)$, and (4.6) by $y^\top F(u) + z^\top H(u)$, and adding the resulting inequalities and with (4.3), we obtain

$$\begin{aligned} y^\top G(u)[(y^\top F)^\circ(u; \eta(x,u)) &+ (z^\top H)^\circ(u; \eta(x,u))] \\ &- [y^\top F(u) + z^\top H(u)](y^\top G)^\circ(u; \eta(x,u)) \geq 0, \quad \text{for all} \quad x \in S. \end{aligned} \tag{4.7}$$

We suppose that

$$\phi(x) < \big(y^\top F(u) + z^\top H(u)\big)/y^\top G(u).$$

Then, by Lemma 2.3 and $y \in U$, we have

$$y^\top F(x)/y^\top G(x) < \big(y^\top F(u) + z^\top H(u)\big)/y^\top G(u).$$

Thus, we have

$$y^\top G(u) y^\top F(x) - y^\top G(x)[y^\top F(u) + z^\top H(u)] < 0.$$

Hence, we have another inequality

$$y^\top G(u)[y^\top F(x) + z^\top H(x)] - y^\top G(x)[y^\top F(u) + z^\top H(u)] < y^\top G(u) z^\top H(x).$$

Using the fact $y^\top G(u) > 0$, $z^\top H(x) \leq 0$, and the latest inequality, we have

$$D(x) < 0 = D(u).$$

Using the fact that $D(\cdot)$ is a pseudoinvex function with respect to η at u, we have

$$\begin{aligned} y^\top G(u)[(y^\top F)^\circ(u; \eta(x,u)) &+ (z^\top H)^\circ(u; \eta(x,u))] \\ &- [y^\top F(u) + z^\top H(u)](y^\top G)^\circ(u; \eta(x,u)) < 0 \end{aligned}$$

which contradicts the inequality (4.7). Hence, the proof is complete. $\square$

Theorem 4.2 (Strong Duality). If x^* is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that (x^*, y^*, z^*) is a feasible solution of (DI). Furthermore, if the conditions of Theorem 4.1 hold for all feasible solutions of (DI), then (x^*, y^*, z^*) is an optimal solution of (DI) and the optimal values of (P) and (DI) are equal; that is, $\min(P) = \max(DI)$.

Proof. By Theorem 3.2, there exist $y^* \in U$, and $z^* \in \mathbb{R}^m_+$, such that (x^*, y^*, z^*) is a feasible solution of (DI). Furthermore,

$$\left(y^{*\top} F(x^*) + z^{*\top} H(x^*)\right) / y^{*\top} G(x^*) = y^{*\top} F(x^*) / y^{*\top} G(x^*) = \phi(x^*).$$

Thus, optimality of (x^*, y^*, z^*) for (DI) follows from Theorem 4.1.

□

Theorem 4.3 (Strict Converse Duality). Let x_1 and (x^*, y_0, z_0) be optimal solutions of (P) and (DI), respectively, and assume that the assumptions of Theorem 4.2 are fulfilled. If

$$D(\cdot) = y_0^\top G(x^*)[y_0^\top F(\cdot) + z_0^\top H(\cdot)] - y_0^\top G(\cdot)[y_0^\top F(x^*) + z_0^\top H(x^*)]$$

is a strictly pseudoinvex function with respect to η, then $x_1 = x^*$; that is, x^* is an optimal solution of (P) with the same optimal values $\phi(x_1) = \left(y_0^\top F(x^*) + z_0^\top H(x^*)\right) / y_0^\top G(x^*)$.

Proof. Suppose, on the contrary, that $x_1 \neq x^*$. From Theorem 4.2 we know that there exist $y_1 \in U$ and $z_1 \in \mathbb{R}^m_+$, such that (x_1, y_1, z_1) is an optimal solution of (DI) and

$$\phi(x_1) = \left(y_1^\top F(x_1) + z_1^\top H(x_1)\right) / y_1^\top G(x_1).$$

Now proceeding as in the proof of Theorem 4.1 (replacing x by x_1 and (u, y, z) by (x^*, y_0, z_0)), we arrive at the following strict inequality:

$$\phi(x_1) > \left(y_0^\top F(x^*) + z_0^\top H(x^*)\right) / y_0^\top G(x^*).$$

This contradicts the fact that

$$\phi(x_1) = \left(y_1^\top F(x_1) + z_1^\top H(x_1)\right) / y_1^\top G(x_1) = \left(y_0^\top F(x^*) + z_0^\top H(x^*)\right) / y_0^\top G(x^*).$$

Therefore, we conclude that

$$x_1 = x^*, \qquad \text{and} \quad \phi(x_1) = \left(y_0^\top F(x^*) + z_0^\top H(x^*)\right) / y_0^\top G(x^*).$$

□

5. SECOND DUAL MODEL

We shall continue our discussion of parameter-free duality model for (P) in this section by showing that the following problem (DII) is also dual problem for (P):

$$(DII) \quad \text{Maximize} \quad y^\top F(u)/y^\top G(u)$$

$$\text{subject to} \quad 0 \in y^\top G(u)\Big(\partial(y^\top F)(u) + \partial(z^\top H)(u)\Big) - y^\top F(u)\partial(y^\top G)(u), \tag{5.1}$$

$$z^\top H(u) \geq 0, \tag{5.2}$$

$$y \in U, \; z \in \mathbb{R}^m_+. \tag{5.3}$$

We denote by K_2 the set of all feasible solutions $(u, y, z) \in X_0 \times U \times \mathbb{R}^m_+$ of problem (DII). Throughout this section, we assume that $y^\top F(u) \geq 0$ and $y^\top G(u) > 0$. Then, we can prove the following weak duality, strong duality, and strict converse duality theorems.

Theorem 5.1 (Weak Duality). Let $x \in S$ and $(u, y, z) \in K_2$ and let

$$E(\cdot) = y^\top G(u) y^\top F(\cdot) - y^\top F(u) y^\top G(\cdot),$$

$$I(\cdot) = z^\top H(\cdot), \qquad \text{and} \qquad J(\cdot) = E(\cdot) + y^\top G(u) I(\cdot).$$

If any one of the following conditions holds

(a) E is a pseudoinvex function with respect to η at u and I is a quasiinvex function at u with respect to same function η,
(b) E is a quasiinvex function with respect to η at u and I is a strictly pseudoinvex function at u with respect to same function η,
(c) J is a pseudoinvex function with respect to η at u.

Then

$$\phi(x) \geq y^\top F(u)/y^\top G(u).$$

Theorem 5.2 (Strong Duality). If x^* is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$ and $z^* \in \mathbb{R}^m_+$, such that (x^*, y^*, z^*) is a feasible solution of (DII). Furthermore, if the conditions of Theorem 5.1 hold for all feasible solutions of (DII), then (x^*, y^*, z^*) is an optimal solution of (DII) and the optimal values of (P) and (DII) are equal; that is, $\min(P) = \max(DII)$.

Theorem 5.3 (Strict Converse Duality). Let x_1 and (x^*, y_0, z_0) be optimal solutions of (P) and (DII), respectively, and assume that the assumptions of Theorem 5.2 are fulfilled. If $E(\cdot) = {y_0}^\top G(x^*) {y_0}^\top F(\cdot) - {y_0}^\top F(x^*) {y_0}^\top G(\cdot)$ is a strictly pseudoinvex function with respect to η and $I(\cdot) = {z_0}^\top H(\cdot)$ is a quasiinvex function with respect to same function η, then $x_1 = x^*$; that is, x^* is an optimal solution of (P) with the same optimal values $\phi(x_1) = y_0^\top F(x^*)/y_0^\top G(x^*)$.

6. THE THIRD DUAL MODEL

Making use of Theorem 3.1, in this section we can formulate the following parametric dual problem:

$$
\begin{aligned}
(DIII)\ \text{Maximize}\ & v \\
\text{subject to}\ & 0 \in \partial(y^\top F)(u) - v\partial(y^\top G)(u) + \partial(z^\top H)(u), && (6.1)\\
& y^\top F(u) - vy^\top G(u) \geq 0, && (6.2)\\
& z^\top H(u) \geq 0, && (6.3)\\
& y \in U,\ \ v \in \mathbb{R}_+, z \in \mathbb{R}^m_+. && (6.4)
\end{aligned}
$$

We denote by K_3 the set of all feasible solutions $(u, y, z, v) \in X_0 \times U \times \mathbb{R}^m_+ \times \mathbb{R}_+$ of problem (DIII). Then a weakly duality theorem is established as follows:

Theorem 6.1 (Weak Duality). Let $x \in S$ and $(u, y, z, v) \in K_3$, and let

$$L(\cdot) = y^\top F(\cdot) - vy^\top G(\cdot),$$

$$I(\cdot) = z^\top H(\cdot), \qquad \text{and} \qquad M(\cdot) = L(\cdot) + I(\cdot).$$

If any one of the following conditions holds

(a) L is a pseudoinvex function with respect to η at u and I is a quasiinvex function at u with respect to same function η,

(b) L is a quasiinvex function with respect to η at u and I is a strictly pseudoinvex function at u with respect to same function η,

(c) M is a pseudoinvex function with respect to η at u.

Then

$$\phi(x) \geq v.$$

Theorem 6.2 (Strong Duality). If x^* is an optimal solution of (P) and that the constraint of (P) satisfy Slater's constraint qualification [8]. Then there exist $y^* \in U$, $z^* \in \mathbb{R}^m_+$, and $v^* \in \mathbb{R}_+$, such that (x^*, y^*, z^*, v^*) is a feasible solution of (DIII). Furthermore, if the conditions of Theorem 6.1 hold for all feasible solutions of (DIII), then (x^*, y^*, z^*, v^*) is an optimal solution of (DIII) and the optimal values of (P) and (DIII) are equal; that is, $\min(P) = \max(DIII)$.

Theorem 6.3 (Strict Converse Duality). Let x_1 and (x^*, y_0, z_0, v_0) be optimal solutions of (P) and (DIII), respectively, and assume that the assumptions of Theorem 6.2 are fulfilled. If $y_0{}^\top F(\cdot) - v_0 y_0{}^\top G(\cdot)$ is a strictly pseudoinvex function with respect to η and $I(\cdot) = z_0{}^\top H(\cdot)$ is a quasiinvex function with respect to same function η, then $x_1 = x^*$; that is, x^* is an optimal solution of (P) with the same optimal values $\phi(x_1) = v_0$.

The complete proof of Theorems 5.1-5.3 and Theorems 6.1-6.3 will be appear elsewhere.

7. SOME REMARKS FOR FURTHER DEVELOPMENTS

(1) There some questions arise that whether the results develop in this paper hold in generalized (F,ρ)-convex ?

(2) Does the set $I = \{1, 2, \cdots, p\}$ in the minimax fractional programming (P) can be replaced by a compact subset Y of $\mathbb{R}^m$? that is, does one can discuss the following minimax fractional programming:

$$\begin{aligned} &\text{Minimize} \quad F(x) = \sup_{y \in Y} \frac{f(x,y)}{g(x,y)} = \sup_{y \in Y} \Psi(x,y) \\ &\text{subject to} \quad h(x) \leq 0, \end{aligned}$$

where Y is a compact subset of $\mathbb{R}^m$?

(3) Do we can discuss this minimax fractional programming in two person game theory ?

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